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Probabilistic Bootstrap Percolation

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In bootstrap percolation, sites are occupied with probability p, but those with less than m occupied first neighbors are removed. This culling process is repeated until a stable configuration (all occupied sites have at least m occupied first neighbors or the whole lattice is empty) is achieved. For $m \geqslant m_1$, the transition is first order, while for $m < m_1$ it is second order, with m-dependent exponents. In probabilistic bootstrap percolation, sites have probability r or $(1-r)$ of being m- or m'-sites, respectively (m-sites are those which need at least m occupied first neighbors to remain occupied). We have studied the model on Bethe lattices, where an exact solution is available. For $m=2$ and $m'=3$, the transition changes from second to first order at $r_1 = 1/2$, and the exponent β is different for $r < 1/2$, $r = 1/2$, and $r > 1/2$. The same qualitative behavior is found for $m=1$ and $m'=3$. On the other hand, for $m=1$ and $m'=2$ the transition is always second order, with the same exponents of $m=1$, for any value of $r > 0$. We found, for $m = z-1$ and $m'=z$, where z is the coordination number of the lattice, that $p_c = 1$ for a value of r which depends on z, but is always above zero. Finally, we argue that, for bootstrap percolation on real lattices, the exponents ν and β for $m=2$ and $m=1$ are equal, for dimensions below 6.

KEY WORDS: Correlated randomness; bootstrap percolation; phase transition; critical exponents; Bethe lattice.

1. INTRODUCTION

In recent years, much effort has been spent on the study of systems with correlated disorder.[1,2] The present scenario is as follows. Systems with short-ranged correlations are in the same universality class as their uncorrelated counterparts. Examples are high-density percolation[3] and site-bond correlated models.[4] On the other hand, long-ranged correlations

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may lead to quite diverse critical behavior. Depending on how fast correlations decay with distance, new exponents and, in some cases, first-order transitions are found.[1,2] Many models exhibit long-ranged correlation effects, but, among them, our interest is directed to the so-called bootstrap percolation (BP) problem,[3] where all the features cited above are present.

In this model, sites on a lattice are randomly occupied with probability p, but those with less than m occupied first neighbors are rendered unoccupied; this culling process is repeated until a stable configuration is achieved (the whole lattice is empty or all occupied sites have at least m occupied first neighbors). The usual percolation problem[6] is reobtained for $m=0$; the case $m=1$ is equivalent to $m=0$, since only isolated sites (which do not contribute to the critical behavior) are culled (from now on both $m=0$ and $m=1$ will be referred as $m=1$ or as usual percolation). The critical concentration p_c (the value of p below which no infinite clusters exist) is the same for $m=1$ and $m=2$; the reason is that, in the latter case, only parts of the dangling ends of the infinite cluster are removed. The backbone of the infinite cluster remains the same as for $m=1$, thus allowing for connection between extremes of the lattice. We will comment further on this topic, when discussing the exponents for $m=2$. For $m\geqslant 3$ critical exponents and probability are lattice dependent; for $m\geqslant m_1$ the transition is first order, where m_1 equals 3 for the square lattice and equals 4 for triangular and simple cubic lattices. For all Bravais lattices studied, all first-order transitions happen at $p_c=1$, which is the exact critical concentration for $m=4$ and 5 on triangular and simple cubic lattices and for $m=3$ on the square lattice.[5,7,8] For these values of m, the approach to the thermodynamic limit, e.g., in numerical simulations, does not follow the usual finite-size scaling prediction[9]:

$$p_c(L)-p_c(\infty)\sim L^{-1/\nu} \tag{1}$$

where L is the size of the (finite) system studied. Instead, the approach to the infinite-size limit has a logarithmic or double-logarithmic dependence on L.[5] With regard to second-order transitions for $m>2$, the results suggest the following. For $m=3$ on the triangular lattice, $p_c=0.628\pm 0.001$ (well above the exact value $p_c=1/2$ for $m=1$[10]), and the numerically calculated exponents seem to be the same as for usual percolation,[11] although more precise evaluations are needed. Recent results for $m=3$ on the simple cubic lattice[12] point to a new behavior for the exponents: while ν is the same as in usual percolation, β is above the value for $m=1$. For Bethe lattices, exact calculations show that the transition is first order for

[3] See ref. 5 for a review on bootstrap percolation and for a comprehensive bibliography on the subject and related models.

$m \geqslant 3$ with $\beta = 1/2$, while for $m = 1$ and 2, the transition is second order and $\beta = 1$ and 2, respectively[13] (these results hold for any value of the coordination number z of the Bethe lattice). We would like to stress that, for percolation, Bethe-lattice exponents hold at and above the upper critical dimension $d_u = 6$.

The motivation for the introduction of BP models comes from a variety of physical systems, such as fluid flow in porous media,[14] crystal field effects in magnetic materials,[13] etc. In the second case, the connection to BP is made through the Blume–Capel Hamiltonian[15]; in this model, the crystal field Δ leads to a nonmagnetic ground state and a first excited ferromagnetic state, with difference in energy between these states proportional to Δ. In a mean-field picture, exchange interactions J can be seen as magnetic fields, which split the excited state, thus changing the ground state to a ferromagnetic one, for suitable ratios Δ/zJ, where z is the coordination number of the lattice. Assuming this to be the picture, dilution (replacement of potentially magnetic atoms by nonmagnetic ones) diminishes the influence of the exchange interaction, through the value of z. So, one can infer that only potentially magnetic atoms with a minimum number m of magnetic first neighbors will develop a magnetic moment. This dynamics is modeled, at zero temperature, by the BP model.

Now, suppose that the crystal field, instead of being uniform, follows a random distribution:

$$\mathscr{P}(\Delta_i) = r\delta(\Delta_i - \Delta) + (1-r)\,\delta(\Delta_i - \Delta') \tag{2}$$

where Δ_i is the crystal field acting on site i and $\delta(x)$ is the Dirac delta function. Thus, the number of magnetic first neighbors necessary to change a given potentially magnetic atom into a magnetic one will depend on the value of Δ_i: atoms on which a crystal field Δ (Δ') acts will need m (m') magnetic first neighbors. A more appropriate model to describe this behavior at zero temperature is the so-called probabilistic bootstrap percolation (PBP).[4] In this model, sites on a lattice are randomly occupied with probability p, but only those with a given minimum number of occupied first neighbors will remain occupied, this number depending on the site through a distribution analogous to Eq. (2). In other words, each site has a probability r or $(1-r)$ of being an m- or m'-site, respectively (m-sites being defined as those which need at least m occupied first neighbors to remain occupied). Our motivation to introduce this model is to study the crossover between integer values of m; this can be done by tuning the value of r from 0 to 1. To the best of our knowledge, this question has not yet been addressed. It may allow, for instance, a first-order

[4] This name was proposed by D. Stauffer in a private communication.

phase transition with $p_c \neq 1$, for $m=3$ and $m'=4$ on the triangular lattice. In this paper we study the PBP on Bethe lattices, where an exact solution is available; critical exponents calculated in this lattice hold at dimensions $d \geqslant d_u$ and, moreover, may give important hints about the critical behavior on Bravais lattices.

This paper is organized as follows. In next section, we discuss the formalism to solve the model on Bethe lattices, and present and discuss the results. In the last two sections we discuss the exponents for $m=2$ and summarize the results.

2. FORMALISM AND RESULTS

The fact that on Bethe lattices z *independent* branches emerge from each site allows for an exact solution of usual percolation[6] and BP models[13] as well as for a variety of other models. The solution for PBP follows closely the ones for the two models cited above. We begin by defining R as the probability that a given occupied site A is *not* connected to an infinite cluster through its first neighbor B. Recalling that B has probability r of being an m-site and probability $(1-r)$ of being an m'-site, one can write

$$R = 1 - p + p\left\{ r \sum_{n=0}^{m-2} C_{z-1}^{n} R^{(z-1-n)}(1-R)^{n} + (1-r) \sum_{n=0}^{m'-2} C_{z-1}^{n} R^{(z-1-n)}(1-R)^{n} \right\} \tag{3}$$

$m, m' = 2, 3, \ldots, z-1, z$.

This equation can be understood as follows: A is not linked to an infinite cluster containing B if B is not present, or if B is present and is an m-site (probability r) but has less than m occupied first neighbors, or, finally, if B is present and is an m'-site (probability $1-r$) but has less than m' occupied first neighbors. This equation holds because only infinite clusters are allowed on the Bethe lattice, for $m \geqslant 2$. We note that the $m=1$ equation for R is the same as for $m=2$, namely

$$R = 1 - p + pR^{(z-1)} \tag{4}$$

The probability that a site belongs to an infinite cluster is given by

$$P(p) = p\left\{ r \sum_{n=0}^{z-m} C_{z}^{n} R^{n}(1-R)^{(z-n)} + (1-r) \sum_{n=0}^{z-m'} C_{z}^{n} R^{n}(1-R)^{(z-n)} \right\} \tag{5}$$

m, $m' = 1, 2, \ldots, z-1, z$; i.e., the site must be occupied and have at least m (m') occupied first neighbors in the case it is an m- (m'-) site. For r and m equals to 1, we regain the usual percolation result, while for $r = 1$ we regain BP.

In order to obtain $P(p)$, we have to solve Eq. (3), which is of order $z-1$. Since $R = 1$ is always a solution, it is convenient to express it in the form $0 = (1-R) f(R, p, r)$ and solve for

$$f(R, p, r) = 0 \tag{6}$$

The solution R of the latter equation is then used to calculate $P(p)$ through Eq. (5). The dependence of $P(p)$ on p near p_c is defined through

$$P(p) - P(p_c^+) \sim A(p - p_c)^{\beta}, \qquad p \to p_c^+ \tag{7}$$

For second- (first-) order transitions, $P(p_c^+)$ equals (is above) 0.

For $m = 1$ and $m' = 2$, Eq. (3) reduces to Eq. (4); so it turns out that $p_c = 1/(z-1)$ for any value of $1 \geqslant r \geqslant 0$, which is the expected result. The transition is always second order and the exponent β is given by

$$\beta = \begin{cases} 1, & r > 0 \\ 2, & r = 0 \end{cases} \tag{8}$$

with an amplitude A proportional to r, for $r \neq 0$. The fact that for any value of $r > 0$ the exponents are the same as for $r = 1$ is due to the presence of dead ends in the infinite cluster for $r > 0$. The number of sites in the infinite cluster decreases as $r \to 0$, but this is reflected in the behavior of the amplitude A. Figure 1 shows $P(p)$ as a function of p for $m = 1$, $m' = 2$. In the following section we will comment on the expected behavior for these values of m and m' on Bravais lattices for dimensions below 6.

For $m = 2$ and $m' = 3$, we expect that the transition changes from second to first order at an intermediate value of r: we call this value r_1. It can be calculated as follows. From the $z-1$ possible solutions to Eq. (6), the physical one satisfies $R(p=1) = 0$. For first-order transitions, this solution holds from $p = 1$ down to p_c, where $R(p_c) \neq 1$; below this value of p, the only physical solution is $R = 1$. Also, the derivative dR/dp diverges at $p = p_c$ for first-order transitions.[13] On the other hand, for second-order transitions $R(p_c) = 1$ and dR/dp is finite at p_c. The border between these two behaviors is signaled by a divergence of dR/dp at $R = 1$. Imposing this on Eq. (6), we find $r_1 = 1/2$, for any value of z. It is also easy to show that

$$dR/dp \sim -(p - p_c)^{-1/2}, \qquad p \to p_c^+, \qquad r = 1/2 \tag{9}$$

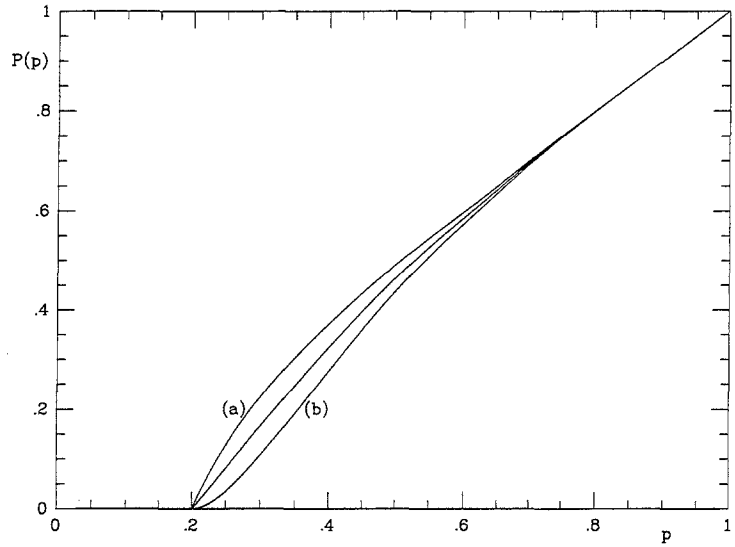

Fig. 1. Order parameter $P(p)$ for PBP with $m=1$ and $m'=2$, on a Bethe lattice with $z=6$. Curves (a) and (b) represent $r=1$ and $r=0$, respectively, and the curve in the middle represents $r=1/2$.

For $r \geqslant r_1$, the critical concentration p_c is given by Eq. (6) with $R=1$:

$$p_c = 1/r(z-1), \qquad \forall z \tag{10}$$

We give p_c for some values of r below r_1 in Table I: as expected, p_c increases for decreasing r. For the exponent β we obtain

$$\beta = \begin{cases} 2, & 1 \geqslant r > 1/2 \\ 1, & r = 1/2 \\ 1/2, & 1/2 > r \geqslant 0 \end{cases} \tag{11}$$

Table I. Values of $p_c(r)$, $R(p_c^+, r)$, and $P(p_c^+, r)$ for PBP on a Bethe Lattice with $z=6$, $m=2$, and $m'=3$[a]

r	p_c	$R(p_c^+, r)$	$P(p_c^+, r)$
1/2	0.40	1.00	0.00
1/3	0.50	0.78	0.11
1/4	0.53	0.71	0.17
1/5	0.55	0.68	0.20
0	0.60	0.58	0.29

[a] A similar table for $m=1$ and $m'=3$ would change only in the values for $P(p_c^+, r)$.

Note that, although the transition is second order for $r = r_1$, β is different from the one obtained for $r > 1/2$ [see Fig. 2 for a picture of $P(p)$ for this case].

We have also studied the PBP for $m = 1$ and $m' = 3$. All results related to the values of p_c and r_1 are the same as for $m = 2$ and $m' = 3$, namely $p_c = 1/r(z-1)$ and $r_1 = 1/2$. The reason is that Eq. (3) is the same for $m = 1$ and $m = 2$ [cf. Eq. (4)]. This equivalence does not hold for Eq. (5), so the exponent β has a different behavior:

$$\beta = \begin{cases} 1, & 1 \geqslant r > 1/2 \\ 1/2, & r = 1/2 \\ 1/2, & 1/2 > r \geqslant 0 \end{cases} \tag{12}$$

For $r = r_1$, β is the same as for $1/2 > r \geqslant 0$, but the transition is second order (see Fig. 3).

Finally, we studied the case $m = z - 1$ and $m' = z$. The transition is always first order with $\beta = 1/2$, as expected, and the value of p_c increases with $1 - r$, reaching the upper limit 1 (the critical concentration for BP with $m = z$) at a value of r, r_0, given by

$$r_0 = (z-2)/(z-1) \tag{13}$$

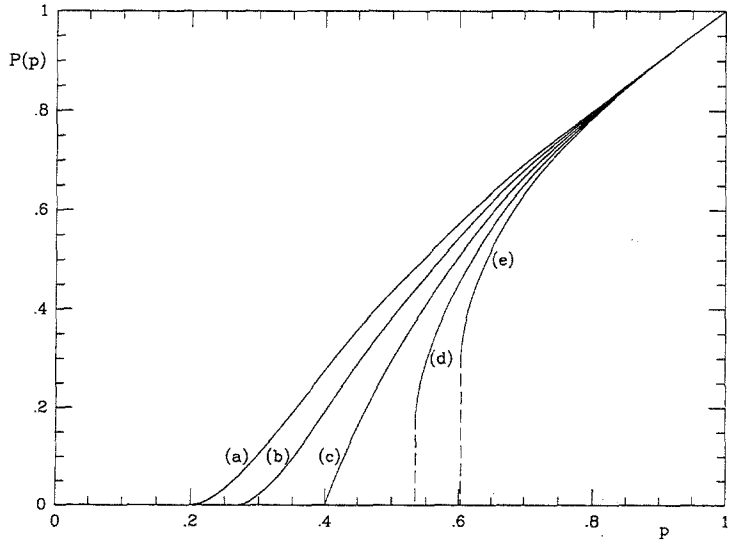

Fig. 2. Order parameter $P(p)$ for PBP with $m = 2$ and $m' = 3$, on a Bethe lattice with $z = 6$, for the following values of r: (a) 1, (b) 3/4, (c) 1/2, (d) 1/4, and (e) 0.

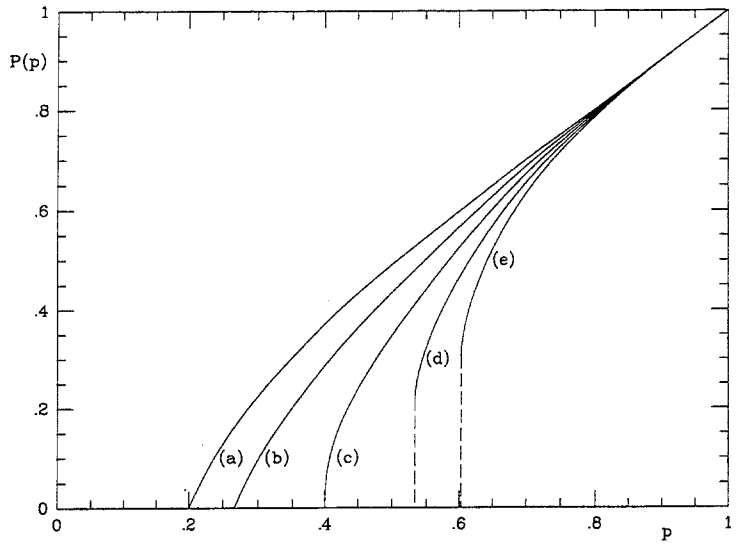

Fig. 3. Same as for Fig. 2, for $m=1$ and $m'=3$.

So, $p_c=1$ is always obtained for a value of m below z. In particular, note that $r_0 \to 1$ as $z \to \infty$; this is expected, since, for large values of z, the probability that a site has at least $z-1$ first neighbors occupied is given by $p^z + zp^{(z-1)}(1-p)$, which is negligibly small for $p \neq 1$.

3. BRAVAIS LATTICES

In this section we briefly outline the possible critical behavior for PBP on Bravais lattices.

For $m=1$ and $m'=2$ on Bravais lattices, we do not expect the same picture as on Bethe lattices, since it can be conjectured that, for BP on dimensions below 6, the $m=2$ case is in the same universality class as usual percolation. The argument goes as follows. With regard to the exponent ν, Adler and Stauffer[12] argued that $\nu_2=\nu_1$, where the subscripts refer to the values of m. The reasoning is the same as that applied by Coniglio,[16] who showed that ν is solely determined by the number of cutting bonds in the backbone of the infinite cluster (cutting bonds are those that, if at least one is cut, the cluster is broken into two disconnected parts). Since this number is the same for $m=1$ and $m=2$ (see below), $\nu_2=\nu_1$ for all dimensions d. On the other hand, it is expected that $\beta_1 \leqslant \beta_2 \leqslant \beta_{bb}$, where β_{bb} is the exponent for the backbone problem[2,12] (at and above $d=6$, the Bethe lattice result holds and $\beta_2=\beta_{bb}$). The reason is that the infinite cluster for $m=2$

is larger than for the backbone problem but smaller than for $m=1$. As discussed in the Introduction, the difference between the infinite clusters for $m=1$ and $m=2$ is that, for the latter, a fraction of the cutting bonds (those at the end of the dangling structures) is removed. In ref. 16 it is shown that, in dangling ends of size $l \ll \xi$ (correlation length), the number of cutting bonds goes to zero at p_c, while for dangling ends such that $l \sim \xi$, the number of cutting bonds, N_{cb} diverges as $|p-p_c|^{-1}$, leading to $N_{cb} \sim L^{1/\nu}$. On the other hand, the number of sites in the infinite cluster scales as $N \sim L^{d-\beta/\nu}$, where L is the size of the system. Since $1/\nu < (d-\beta/\nu)$ for $2 \leqslant d < 6$, the number of cutting bonds is a set of null measure, compared to the number of sites in the infinite cluster. As the cutting bonds at the extremes of the dangling ends form a subset of the total number of cutting bonds, the culling process for $m=2$ will remove an irrelevant fraction of the sites and we expect $\beta_2 = \beta_1$. Simulations in two[11] and three[12] dimensions support this picture. So PBP on Bravais lattices for $m=1$ and $m'=2$ is trivial, in the sense that for all values of r the critical behavior is the same.

Nevertheless, new features are expected for $m=1$ and $m'>3$. Since for the former the transition is second order, while for the latter it is first order with $p_c=1$, a picture similar to the case $m=1$ and $m'=3$ on Bethe lattices may arise; one may find first-order transitions with $p_c \neq 1$ and a different universality class for some intermediate value of r. However, one cannot exclude the possibility that $r_1 = r_0$, in which case there would be no range of r such that the transition is first order but $p_c \neq 1$. Another point of interest is how the "large-void-instabilities argument"[7,18,13] would change for PBP: this can give a hint on the existence of a new universality class for intermediate values of r as well as for the value of r_1. Studies are now being carried out using real-space renormalization group (RSRG) procedures and Monte Carlo simulation. While the former is convenient for studying universality classes, the latter is more adequate for an accurate evaluation of critical exponents.

4. SUMMARY

We have studied the PBP model on Bethe lattices: when $m'=3$ is involved a new critical point is found for $1>r>0$, which is the border between second- and first-order transitions. In the RSRG context, this critical point is an unstable fixed point. This behavior may be present, for instance, in the PBP on the triangular lattice, for $m=1$ and $m'=4$ (where it is necessary that $r_1 \neq r_0$, in order to observe first-order transitions with $p_c \neq 1$). For $m=1$ and $m'=2$, the exponent β is the same as for $m=1$ for any value of r above zero. This behavior is explained by the presence of

finite clusters for any $r > 0$. For $m = z - 1$ and $m' = z$, r_0 (the value of r for which $p_c = 1$) is always above zero. Finally, we argue, based on the fractal dimension of the number of cutting bonds, that BP with $m = 2$ is in the same universality class as usual percolation.

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